MODEL TEST 5 - REZOLVĂRI

1. $C.E.\begin{cases} x\geq 4 \\ x+1\geq 3 \\ x\in\mathbb{N}\end{cases} \Rightarrow \begin{cases} x\in(-\infty,-1)\cup(1,+\infty) \\ x\in(-\infty,3) \\ x\in(-4,+\infty), x\neq -3\end{cases} \Rightarrow$

$\Rightarrow D_{C.E}=\{x\in\mathbb{N}; x\geq 4\}.$

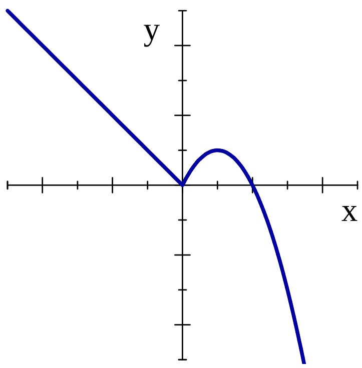
Atunci

$23\dfrac{x!}{(x-4)!}=24\left[\dfrac{(x+1)!}{(x-2)!}-\dfrac{x!}{(x-4)!\cdot 4!}\right]\Leftrightarrow$

$23+1=24\dfrac{x+1}{(x-3)(x-2)}\Leftrightarrow x^2-6x+5=0.$

Răspuns corect (c).

2.



Din grafic rezultă că f este surjectivă pe $\mathbb{R}$, nu este injectivă pe $\mathbb{R}$, nu este monotonă pe $\mathbb{R}$.

Răspuns corect (a).

3. $a_n=\dfrac{-1}{7}\cdot\dfrac{\left(\frac{-1}{7}\right)^n-1}{\frac{-1}{7}-1}\Rightarrow$

$l_1=\lim\limits_{n\to\infty}a_n=\dfrac{-1}{7}\cdot\dfrac{0-1}{\frac{-1}{7}-1}=-\frac{1}{8}.$

$b_n=\sum\limits_{k=1}^{n}\dfrac{k}{(k+1)!}=\sum\limits_{k=1}^{n}\left[\dfrac{1}{k!}-\dfrac{1}{(k+1)!}\right]=1-\dfrac{1}{(n+1)!}\Rightarrow$

$l_2=\lim\limits_{n\to\infty}b_n=1-0=1.$

Răspuns corect (c).

4. Dacă $x\leq 2,016\Rightarrow$

$-x+2,016\leq 3-x\Rightarrow 2,016\leq 3$ Se verifică, $\forall x\leq 2,016$.

Dacă $x>2,016\Rightarrow$

$x-2,016\leq 3-x\Rightarrow 2x\leq 5,016\Rightarrow x\leq 2,508.$

Deci $A=(-\infty,2,016]\cup(2,016,2,508]=(-\infty,2,508]$.

În A sunt următoarele numere întregi mai mari decât $-2,016$: $-2,-1,0,1,2$.

Răspuns corect (a).

5. $E = x_1x_2\left[(x_1+x_2)^2 - 2x_1x_2\right] =$

$= 16 \cdot \left((-20)^2 - 2 \cdot 16\right) = 5888$

6. C simetricul lui A faţă de $B \Rightarrow B$ este mijlocul segmentului $[AB] \Rightarrow$

$$\begin{cases} 3 = \dfrac{1+x_C}{2} \\ 4 = \dfrac{2+y_C}{2} \end{cases} \Rightarrow C(5,6).$$

Dreapta care trece prin A şi B are panta $m_{AB} = \dfrac{4-2}{3-1} = 1 \Rightarrow$

Perendiculara pe AB are panta $m_d = -1$.

Dreapta din enunţ are ecuaţia:

$(d): y - 6 = -1(x-5) \Leftrightarrow$

$(d): x + y - 11 = 0$

Răspuns corect (d).

7. Fie $p = i \cdot i^2 \cdot ... \cdot i^{100}$. Să se determine $a \in \mathbb{R}$ astfel încât funcţia $f : [0, 2000] \to \mathbb{R}, f(x) =$
$$\begin{cases} a \cdot p \cdot \cos(x-1), & \text{dacă } x \in [0,1] \\ \dfrac{\sqrt{x^2+8}-3}{x^2-3x+2} & \text{dacă } x \in (1, 2000] \end{cases}$$
să fie continuă pe $[0, 2000]$.

(a) $a = \dfrac{-1}{3}$; **(b)** $a = \dfrac{1}{3}$; **(c)** $a = 0$; **(d)** nu există a cu proprietatea cerută.

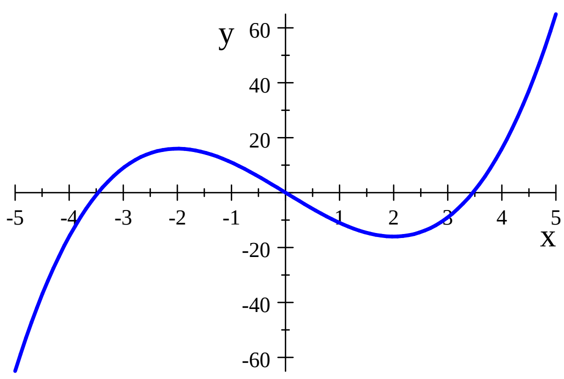
8.$\Delta = -3m^2 + 6m + 1$.

$\Delta < 0 \Leftrightarrow -3m^2 + 6m + 1 < 0 \Leftrightarrow 3m^2 - 6m - 1 > 0 \Leftrightarrow$

$\Leftrightarrow m \in \left(-\infty, 1 - \frac{2\sqrt{3}}{3}\right) \cup \left(1 + \frac{2\sqrt{3}}{3}, +\infty\right)$.

Răspuns corect (d).

9..Graficul funcţiei $f : \mathbb{R} \to \mathbb{R}, f(x) = x^3 - 12x$ este



$f'(x) = 3x^2 - 12 = 3\left(x^2 - 4\right)$

$f''(x) = 6x$

x	$-\infty$ $\quad$ -2 $\quad$ 0 $\quad$ 2 $\quad$ $+\infty$
$f'(x)$	$+\ +\ +\ +\ 0\ -\ -\ -\ -\ -\ -\ -\ 0\ +\ +\ +\ +\ +$
$f''(x)$	$-\ -\ -\ -\ -\ -\ -\ -\ 0\ +\ +\ +\ +\ +\ +\ +\ +\ +$
$f(x)$	$-\infty$ $\nearrow\frown$ 16 $\searrow\frown$ 0 $\searrow\smile$ -16 $\nearrow\smile$ $+\infty$

Puncte de extrem local $(-2, 16)$ şi $(2, -16)$.
Răspuns corect (a).

10. $z = \left[\frac{1}{\sin\frac{(2k+1)\pi}{2n}}\left(\cos\frac{(2k+1)\pi}{2n} + i\sin\frac{(2k+1)\pi}{2n}\right)\right]^n =$

$= \left(\frac{1}{\sin\frac{(2k+1)\pi}{2n}}\right)^n \cdot \left(\cos\frac{(2k+1)\pi}{2} + i\sin\frac{(2k+1)\pi}{2}\right).$

$\cos\frac{(2k+1)\pi}{2} = 0, \forall k \in \{0, 1, ..., n-1\}$
Răspuns corect (c).

11. $C.E. \begin{cases} x^2 - 1 > 0 \\ 6 - 2x > 0 \\ x + 4 > 0, x + 4 \neq 1 \end{cases} \Rightarrow \begin{cases} x \in (-\infty, -1) \cup (1, +\infty) \\ x \in (-\infty, 3) \\ x \in (-4, +\infty), x \neq -3 \end{cases} \Rightarrow$

$\Rightarrow D_{C.E} = (-4, -3) \cup (-3, -1) \cup (1, 3)$
$\log_{x+3}\left(x^2 - 1\right) = \log_{x+3}(6 - 2x) \Rightarrow x^2 - 1 = 6 - 2x \Rightarrow$
$\Rightarrow x^2 + 2x - 3 = 0 \Rightarrow x_1 = -3 \notin D_{C.E}$ şi $x_2 = 1 \notin D_{C.E}$.
Răspuns corect (d).

12. Fie $g : \mathbb{R} \to \mathbb{R}, g(t) = \ln\left(1 + \sin^2 t\right)$. Deoarece g este continuă pe $\mathbb{R}$, funcţia admite primitive pe $\mathbb{R}$, chiar dacă acestea nu pot fi exprimate cu funcţii elementare. Fie G o astfel de primitivă. Atunci
$f(x) = G(\arcsin x) - G\left(\frac{\pi}{4}\right), \forall x \in (-1, 1) \Rightarrow$
$f'(x) = [G'(\arcsin x)](\arcsin x)' - 0 =$

$= [g(\arcsin x)]\frac{1}{\sqrt{1-x^2}} = \frac{\ln\left(1+x^2\right)}{\sqrt{1-x^2}}, \forall x \in (-1, 1).$

Răspuns corect (b).

13. $x_1 = \frac{7}{1-\sqrt{8}} = \frac{7\left(1+\sqrt{8}\right)}{1^2 - \left(\sqrt{8}\right)^2} = -1 - \sqrt{8}$

$x_2 = \frac{10}{1-3i} = \frac{10(1+3i)}{1^2 - (3i)^2} = 1 + 3i$

Atunci P are ca rădăcini şi
$x_3 = -1 + \sqrt{8}$
$x_4 = 1 - 3i$
$P(X) = \left(X^2 + 2X - 7\right) \cdot \left(X^2 - 2X + 10\right) = X^4 - X^2 + 34X - 70$
Răspuns corect (d).

14. $A(2, 1) \in (d_1) \Rightarrow 2 \cdot 2 + a \cdot 1 - 7 = 0 \Rightarrow a = 3 \Rightarrow$
$\Rightarrow (d_1) : 2x + 3y - 7 = 0.$
$B(0, 4) \in (d_2)$ şi $C(6, 0) \in (d_2) \Rightarrow$

$\Rightarrow (d_2) : \dfrac{x-0}{6-0} = \dfrac{y-4}{0-4} \Rightarrow$
$\Rightarrow (d_2) : 4x + 6y - 24 = 0 \Rightarrow$
$\Rightarrow (d_2) : 2x + 3y - 12 = 0.$
Dreptele (d_1) şi (d_2) sunt paralele
Răspuns corect (d).

15. $5^{\frac{2\left(r^2-r-\frac{3}{2}\right)}{3}} = 5^3 \Leftrightarrow 2r^2 - 2r - 3 = 9 \Leftrightarrow r^2 - r - 6 = 0 \Rightarrow$
$r = 3 < 0$ şi $r = -2 < 0.$
Progresia aritmetică fiind descrescătoare, se alege $r = -2$
$T_{k+1} = C_{10}^k \left(\sqrt[3]{x^2}\right)^{10-k} \left(\dfrac{2}{\sqrt[4]{x}}\right)^k = C_{10}^k x^{\frac{2(10-k)}{3}} 2^k x^{\frac{-k}{4}}$
$= C_{10}^k 2^k x^{\frac{2(10-k)}{3} + \frac{-k}{4}}$
independenţi de $x \Rightarrow \frac{2(10-k)}{3} + \frac{-k}{4} = 0 \Rightarrow$
$8(10-k) - 3k = 0 \Rightarrow 80 - 11k = 0 \Rightarrow k = \frac{80}{11} \notin \mathbb{N} \Rightarrow h = 0.$
Atunci $a_{10} = 9 - 18 = -9$
Răspuns corect (b).

16. $\dfrac{x_{n+1}}{x_n} = \dfrac{[(n+1)!]^2}{(2n+2)!} \cdot \dfrac{(2n)!}{[n!]^2} = \dfrac{(n+1)^2}{(2n+1)(2n+2)} \Rightarrow$
$l_1 = \lim\limits_{n\to\infty} \dfrac{x_{n+1}}{x_n} = \frac{1}{4}.$
$y_n = \left(1 + \dfrac{1}{1 + \frac{n-2016}{2017}}\right)^{\frac{n}{2017}} = \left(1 + \dfrac{2017}{n+1}\right)^{\frac{n+1}{2017} \cdot \frac{n}{n+1}} \Rightarrow$
$l_2 = \lim\limits_{n\to\infty} y_n = e^1 = e.$
Răspuns corect (c).

17. Echipa se poate forma:
-din 2 studenţi de anul al II-lea şi 3 de anul I în $C_4^2 \cdot C_7^3$ moduri;
-din 3 studenţi de anul al II-lea şi 2 de anul I în $C_4^3 \cdot C_7^2$ moduri;
-din 4 studenţi de anul al II-lea şi 1 de anul I în $C_4^4 \cdot C_7^1$ moduri.
În total sunt un număr de moduri egal cu:
$C_4^2 \cdot C_7^3 + C_4^3 \cdot C_7^2 + C_4^4 \cdot C_7^1 = 210 + 84 + 7 = 301.$
Răspuns corect (b).

18.
$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$
$$\begin{pmatrix} -1 & 0 & 1 \\ 2 & 0 & 0 \\ 0 & 2 & -1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 1 \\ 2 & 0 & 0 \\ 0 & 2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & -2 \\ -2 & 0 & 2 \\ 4 & -2 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 2 & -2 \\ -2 & 0 & 2 \\ 4 & -2 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & -1 \\ -2 & 1 & 0 \\ 4 & -2 & 2 \end{pmatrix}$$
$\det\left(A^2+B^2\right)=4.$
Precizăm că $\det A = 1$ şi $\det B = 4$, dar $\det\left(A^2+B^2\right) \neq (\det A)^2 + (\det B)^2$.
Răspuns corect (a).

19. $\overrightarrow{a} = \overrightarrow{i} + \overrightarrow{j}$ şi $\overrightarrow{b} = \frac{\sqrt{2}}{2}\overrightarrow{i} + m\overrightarrow{j}$ sunt perpendiculari$\Rightarrow$
$1 \cdot \frac{\sqrt{2}}{2} + 1 \cdot m = 0 \Rightarrow m = -\frac{\sqrt{2}}{2}$.
$E = -2 + \cos\frac{\pi}{6} \cdot \sin\frac{\pi}{4} = -\frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{6}-2\sqrt{2}}{4}$.
Răspuns corect (a).

20.
$$B = \begin{pmatrix} 2 & 0 \\ 1 & 6 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 1 & 6 \end{pmatrix} - 8\begin{pmatrix} 2 & 0 \\ 1 & 6 \end{pmatrix} + 13\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$B^{2016} = I_2$
$$A^3 = \begin{pmatrix} 2 & 0 \\ 1 & 6 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 1 & 6 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 1 & 6 \end{pmatrix} = \begin{pmatrix} 8 & 0 \\ 52 & 216 \end{pmatrix}$$
$c = 8 + 216 + 52 = 276$
Răspuns corect (a).

21.
$$\begin{vmatrix} 2 & 1 & 1 \\ 0 & -m & 1 \\ m^2 & 0 & 1 \end{vmatrix} \neq 0 \Leftrightarrow$$
$m^3 + m^2 - 2m \neq 0 \Leftrightarrow m\left(m^2+m-2\right) \neq 0 \Rightarrow M = \mathbb{R}\setminus\{-2,0,1\}$.
Răspuns corect (a).

22. $A = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \dfrac{x}{\cos^2 x}dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} x \cdot (\operatorname{tg} x)'\,dx = x \cdot (\operatorname{tg} x)\Big|_{x=\frac{\pi}{6}}^{x=\frac{\pi}{3}} - \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (\operatorname{tg} x)\,dx =$
$= \dfrac{\pi}{3}\sqrt{3} - \dfrac{\pi}{3} \cdot \dfrac{\sqrt{3}}{3} + \ln(\cos x)\Big|_{x=\frac{\pi}{6}}^{x=\frac{\pi}{3}} = \dfrac{2\pi}{9}\sqrt{3} + \ln\dfrac{1}{2} - \ln\dfrac{\sqrt{3}}{2} =$
$= \dfrac{2\pi}{9}\sqrt{3} - \ln\sqrt{3}.$
Răspuns corect (a).

23.x a.î. $\cos x = 0$ nu verifică ecuaţia. Atunci $\operatorname{tg} x - \sqrt{3} = 0$.
Răspuns corect (b).

24. Legea e asociativă, comutativă
$x * e = x \Rightarrow xe + x + e = x \Rightarrow (x+1)e = 0 \Rightarrow e = 0.$
$z * 2000 = e \Rightarrow 2000z + z + 2000 = 0 \Rightarrow z = \frac{-2001}{2000}$.
Răspuns corect (b).

25. $f' : \mathbb{R} \to \mathbb{R}, f'(x) = \operatorname{arctg} x + x \cdot \dfrac{1}{1+x^2} - \dfrac{2x}{1+x^2} = \operatorname{arctg} x - \dfrac{x}{1+x^2}$;

$f'':\mathbb{R}\to\mathbb{R}, f''(x)=\dfrac{1}{1+x^2}-\dfrac{1+x^2-x\cdot 2x}{\left(1+x^2\right)^2}=\dfrac{2x^2}{\left(1+x^2\right)^2};$

$f''(x)\geq 0, \forall x\in\mathbb{R}\Rightarrow f'$ e crescătoare pe $\mathbb{R}$.

$f'(1)=\operatorname{arctg}1-\dfrac{1}{1+1^2}=\frac{\pi}{4}-\frac{1}{2}.$

$f''(1)=\frac{1}{2}.$

Răspuns corect (a).

26. Fie $f:(0,2)\to\mathbb{R}, f(x)=\dfrac{2}{x^2+2x}$. Atunci $f^{(4)}(1)$ are valoarea:

(a) $4!\left(1-\frac{1}{3^5}\right)$; **(b)** $\frac{242}{243}$; **(c)** $24\left(1+\frac{1}{3^5}\right)$; **(d)** $6\left(1-\frac{1}{3^4}\right)$.

Rezolvare:

$f(x)=\dfrac{2}{x^2+2x}=\dfrac{1}{x}-\dfrac{1}{x+2}=x^{-1}-(x+2)^{-1}.$

$f'(x)=(-1)\,x^{-2}-(-1)(x+2)^{-2}.$

$f''(x)=(-1)(-2)\,x^{-3}-(-1)(-2)(x+2)^{-3}.$

$f'''(x)=(-1)(-2)(-3)\,x^{-4}-(-1)(-2)(-3)(x+2)^{-4}.$

$f^{(4)}(x)=(-1)(-2)(-3)(-4)\,x^{-5}-(-1)(-2)(-3)(-4)(x+2)^{-5}.$

$f^{(4)}(x)=4!\dfrac{1}{x^5}-4!\dfrac{1}{(x+2)^5}$

$f^{(4)}(1)=4!\left(1-\frac{1}{3^5}\right)=24\cdot\frac{242}{243}=\frac{1936}{81}.$

Răspuns corect (d).

27 $\int xe^{-x}dx=-xe^{-x}-e^{-x}+c, \forall x\in(0,+\infty), \forall c\in\mathbb{R}.$

$\int\dfrac{\sqrt{x}}{x+1}dx=?$

$$\begin{cases}\sqrt{x}=t,\ t\in(0,+\infty)|\ \text{inversăm}\\ x=t^2,\ t\in(0,+\infty)|\ \text{diferenţiem}\\ dx=2tdt\end{cases}$$

$\int\dfrac{t}{t^2+1}(2t)\,dt=2\int\dfrac{t^2+1-1}{t^2+1}dt=$

$=2\left(t-\operatorname{arctg}t\right)+\widetilde{c},\ \forall t\in(0,+\infty), \forall\widetilde{c}\in\mathbb{R}.$

$\int\dfrac{\sqrt{x}}{x+1}dx=2\left(\sqrt{x}-\operatorname{arctg}\sqrt{x}\right)+c,\ \forall x\in(0,+\infty), \forall c\in\mathbb{R}.$

Răspuns corect (b).

28 $\displaystyle\int_{2000}^{2001}[x]\,dx=\int_{2000}^{2001}2000dx=2000.$

$\displaystyle\int_0^2\frac{2x-[x]}{x+[x+2]-1}dx=\int_0^1\frac{2x}{x+1}dx+\int_1^2\frac{2x-1}{x+2}dx=5\ln 3-12\ln 2+4.$

Răspuns corect (a).

29. $D(-x_A,-y_A)$ este simetricul lui A faţă de $O(0,0)\Rightarrow|AD|=2|OA|$.

Atunci valoarea minimă a segmentului $[AB]$ se atinge când

$|OA|=dist\left(O,(d)\right)=\frac{|0-2\cdot 0+2|}{\sqrt{5}}=\frac{2}{\sqrt{5}}.$

Răspuns corect (d).

30. $|z| = \sqrt{\left(5\cos u\right)^2 + \left(5\sin u\right)^2} = 5.$
$\frac{4}{3} \in \left(0, \frac{\pi}{2}\right) \Rightarrow$
$\operatorname{Re} z = 5\cos u = 5\cos\left(\operatorname{arctg}\frac{4}{3}\right) = 5\frac{+1}{\sqrt{1+\left(\frac{4}{3}\right)^2}} = 3.$

Răspuns corect (d).